

# A note on static dyonic diholes

H. García-Compeán, V. S. Manko

*Departamento de Física,*

*Centro de Investigación y de Estudios Avanzados del IPN,*

*A.P. 14-740, 07000 México D.F., Mexico*

In this brief note we argue that a dyonic generalization of the Emparan-Teo dihole solution is described by a static diagonal metric and therefore, contrary to the claim made in a recent paper by Cabrera-Munguia et al., does not involve any “non-vanishing global angular momentum” and rotating charges.

PACS numbers: 04.20.Jb, 04.70.Bw, 97.60.Lf

## I. INTRODUCTION

In a recent paper [1], Cabrera-Munguia et al. presented an exact stationary axisymmetric solution of the Einstein-Maxwell equations for two unequal dyons and considered some of its limiting cases. In subsection 5.3 they assert that in the limit of vanishing total angular momentum, their binary dyonic configuration still has “non-vanishing global angular momentum” and, moreover, “the magnetic monopole charges arise from the rotation of the Reissner-Nordström black holes”. Our note aims at demonstrating explicitly that the dyonic generalizations of the symmetric [2] and asymmetric [3] dihole spacetimes (to which supposedly must lead that limit) are static intrinsically, so that the claim by Cabrera-Munguia et al. is erroneous and misleading.

## II. THE DYONIC EMPARAN-TEO DIHOLE

The static dihole solution describing two Reissner-Nordström black holes [4, 5] with equal masses and opposite electric charges was obtained by Emparan and Teo [2] as a particular specialization of the double-Kerr-Newman solution [6]. The physical parametrization of the Emparan-Teo dihole and its magnetostatic analog were later worked out in [7], the two static versions of the dihole spacetime being related by Bonnor’s theorem [8]. The dyonic generalization of the Emparan-Teo dihole is obtainable from the original electrostatic solution by means of the duality rotation, and below we write down the corresponding expressions of the Ernst potentials [9], satisfying the equations (for real  $\mathcal{E}$ )

$$\begin{aligned}(\mathcal{E} + \bar{\Phi}\Phi)\nabla^2\mathcal{E} &= (\nabla\mathcal{E} + 2\bar{\Phi}\nabla\Phi) \cdot \nabla\mathcal{E}, \\(\mathcal{E} + \bar{\Phi}\Phi)\nabla^2\Phi &= (\nabla\mathcal{E} + 2\bar{\Phi}\nabla\Phi) \cdot \nabla\Phi,\end{aligned}\tag{1}$$

which arise as a by-product of the recent paper [10] on stationary diholes:

$$\begin{aligned}\mathcal{E} &= \frac{A-B}{A+B}, \quad \Phi = \frac{QC}{A+B}, \\A &= R^2(M^2 - |Q|^2\tau)(R_+ - R_-)(r_+ - r_-) + 4\sigma^2(M^2 + |Q|^2\tau)(R_+ - r_+)(R_- - r_-) \\&\quad + 2R^2\sigma^2(R_+r_- + R_-r_+), \\B &= 2MR\sigma[(R\sigma - 2M^2)(R_+ + r_-) + (R\sigma + 2M^2)(R_- + r_+)], \\C &= \frac{2R\sigma(R - 2M)}{R^2 - 4\sigma^2}[(R - 2\sigma)(R\sigma + 2M^2)(R_+ - r_-) + (R + 2\sigma)(R\sigma - 2M^2)(R_- - r_+)], \\R_{\pm} &= \sqrt{\rho^2 + \left(z + \frac{1}{2}R \pm \sigma\right)^2}, \quad r_{\pm} = \sqrt{\rho^2 + \left(z - \frac{1}{2}R \pm \sigma\right)^2}.\end{aligned}\tag{2}$$

In the above formulae

$$\begin{aligned}\mathcal{Q} &= Q + i\mathcal{B}, \quad |\mathcal{Q}|^2 = Q^2 + \mathcal{B}^2, \\ \sigma &= \sqrt{M^2 - |Q|^2} \frac{R - 2M}{R + 2M}, \quad \tau = \frac{R^2 - 4M^2}{(R + 2M)^2 + 4|Q|^2},\end{aligned}\tag{3}$$

the real parameters  $M$ ,  $Q$ ,  $\mathcal{B}$  and  $R$  being, respectively, the mass, electric charge, magnetic charge of the upper dyon and the separation distance; the characteristics of the lower constituent are  $M$ ,  $-Q$ ,  $-\mathcal{B}$ . The Weyl cylindrical coordinates  $(\rho, z)$  enter the expressions (2) only through the functions  $R_{\pm}$  and  $r_{\pm}$ .

By setting  $\mathcal{B} = 0$  in (2) and (3), one comes to the Ernst potentials of the Emparan-Teo electrostatic solution [2], and the limit  $Q = 0$  leads to the magnetostatic analog of the Emparan-Teo dihole [7]. Note, that the function  $\mathcal{E}$  in (2) is real, while the electromagnetic potential  $\Phi$  represents a product of the complex constant  $\mathcal{Q}$  and a real function; therefore, the metric defined by these  $\mathcal{E}$  and  $\Phi$  remains static and diagonal. Below we give the corresponding metric functions  $f$  and  $\gamma$  from the Weyl line element

$$ds^2 = f^{-1}[e^{2\gamma}(d\rho^2 + dz^2) + \rho^2 d\varphi^2] - f dt^2, \quad (4)$$

together with the electric  $A_t$  and magnetic  $A_\varphi$  components of the electromagnetic 4-potential,

$$\begin{aligned} f &= \frac{A^2 - B^2 + |\mathcal{Q}|^2 C^2}{(A + B)^2}, \quad e^{2\gamma} = \frac{A^2 - B^2 + |\mathcal{Q}|^2 C^2}{16R^4 \sigma^4 R_+ R_- r_+ r_-}, \quad A_t = -\frac{QC}{A + B}, \quad A_\varphi = \frac{\mathcal{B}(I - zC)}{A + B}, \\ I &= -\frac{2M(R - 2M)}{R^2 - 4\sigma^2} [2R^2(M^2 - \sigma^2)(R_+ r_+ + R_- r_-) + 2\sigma^2(R^2 - 4M^2)(R_+ R_- + r_+ r_-)] \\ &\quad + (R - 2M) \{ 2M[R\sigma(R_+ r_- - R_- r_+) + 2M^2(R_+ r_- + R_- r_+)] \\ &\quad + R\sigma[R\sigma(R_+ + R_- + r_+ + r_-) + 6M^2(R_+ - R_- - r_+ + r_-) + 8MR\sigma] \}, \end{aligned} \quad (5)$$

and one can see in particular that  $A_\varphi$  is defined by a more concise expression than in the paper [7]. Here it is worth noting that the duality rotation of the potential  $\Phi$  leaves the metric unchanged, and so the transformation

$$\Phi' = e^{i\alpha_0} \Phi, \quad \tan \alpha_0 = -\mathcal{B}/Q, \quad (6)$$

would convert the solution (2) into the electrostatic one ( $\text{Im } \Phi' = 0$ ) described by the Weyl metric with the functions  $f$  and  $\gamma$  from (5), thus confirming the staticity of the dyonic Emparan-Teo solution (2).

The absence of any stationary energy flows created by the electric and magnetic charges in the binary dyonic configuration (2) can be easily established by analyzing the associated Poynting vector. In [11] it was demonstrated that the Poynting vector of a stationary axisymmetric electrovac spacetime can have only one non-zero component, and in [12] this  $\varphi$ -component was shown to be defined by the following simple formula:

$$S^\varphi = \frac{\sqrt{f} e^{-2\gamma}}{4\pi\rho} \text{Im}(\bar{\Phi}_{,\rho} \Phi_{,z}). \quad (7)$$

Then, taking into account that  $\bar{\Phi}_{,\rho} \Phi_{,z}$  is a real function in the case of the solution (2), one immediately gets  $S^\varphi = 0$ , which means the absence of frame-dragging effects due to electromagnetic field in the metric (5). The latter metric is therefore static intrinsically.

Let us also mention for completeness that on the upper and lower horizons ( $\rho = 0$ ,  $\frac{1}{2}R - \sigma \leq z \leq \frac{1}{2}R + \sigma$  and  $\rho = 0$ ,  $-\frac{1}{2}R - \sigma \leq z \leq -\frac{1}{2}R + \sigma$ , respectively) the surface gravity  $\kappa^H$  and horizon's area  $S^H$  of the dyonic Emparan-Teo solution are given by the expressions

$$\kappa^H = \frac{R\sigma(R + 2\sigma)}{(R + 2M)^2(M + \sigma)^2}, \quad S^H = \frac{4\pi(R + 2M)^2(M + \sigma)^2}{R(R + 2\sigma)}, \quad (8)$$

while the Ernst potential  $\Phi$  on the upper horizon takes the complex constant value

$$\Phi_{ext}^H = \frac{\mathcal{Q}(M - \sigma)}{|\mathcal{Q}|^2} \quad (9)$$

(on the lower horizon,  $\Phi_{ext}^H$  changes its sign), thus representing a complex extension of the electric potential  $\Phi^H$  from the Smarr mass formula [13]. The generalized mass relation involving the electric and magnetic charges has the form

$$M = \frac{1}{4\pi} \kappa^H S^H + \bar{\mathcal{Q}} \Phi_{ext}^H, \quad (10)$$

and it is verified identically on both horizons since the complex charge  $\mathcal{Q}$ , similar to  $\Phi_{ext}^H$ , changes its sign on the lower horizon.

### III. THE ASYMMETRIC DYONIC DIHOLE

The case of the dyonic asymmetric dihole solution is fully analogous to the previous case of the dyonic Emparan-Teo dihole. The Ernst potentials of that solution are obtainable by means of the substitutions  $Q \rightarrow \mathcal{Q}$ ,  $Q^2 \rightarrow |\mathcal{Q}|^2$  from the

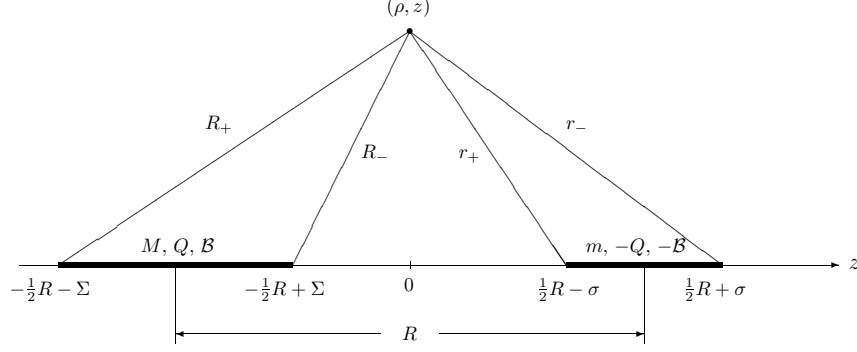


FIG. 1: Location of the dyonic black-hole constituents on the symmetry axis.

formulae of the paper [3] defining the respective potentials of the asymmetric electric dihole, thus eventually leading to the metric functions  $f$  and  $\gamma$  in the Weyl line element (4) and to the electric  $A_t$  and magnetic  $A'_\varphi$  potentials of the form

$$\begin{aligned}
 f &= \frac{A^2 - 4B^2 + 4|\mathcal{Q}|^2 C^2}{(A + 2B)^2}, \quad e^{2\gamma} = \frac{A^2 - 4B^2 + 4|\mathcal{Q}|^2 C^2}{K_0^2 R_+ R_- r_+ r_-}, \quad A_t = -\frac{2QC}{A + 2B}, \quad A'_\varphi = \frac{2\mathcal{B}C}{A + 2B}, \\
 A &= \Sigma \sigma [\nu(R_+ + R_-)(r_+ + r_-) + 4\kappa(R_+ R_- + r_+ r_-)] - (|\mathcal{Q}|^2 \mu^2 \nu - 2\kappa^2)(R_+ - R_-)(r_+ - r_-), \\
 B &= \Sigma \sigma [(m\nu + 2M\kappa)(R_+ + R_-) + (M\nu + 2m\kappa)(r_+ + r_-)] + |\mathcal{Q}|^2 (\mu - \mu^2)(\nu - 2\kappa) \\
 &\quad \times [\sigma(R_+ - R_-) - \Sigma(r_+ - r_-)] - 2R\kappa[M\sigma(R_+ - R_-) - m\Sigma(r_+ - r_-)], \\
 C &= \Sigma \sigma (1 - \mu)(\nu - 2\kappa)(r_+ + r_- - R_+ - R_-) + \sigma[M\mu\nu + 2\kappa(m\mu - R + R\mu)](R_+ - R_-) \\
 &\quad + \Sigma[m\mu\nu + 2\kappa(M\mu - R + R\mu)](r_+ - r_-), \\
 K_0 &= 4\Sigma \sigma [R^2 - (M - m)^2 + 4|\mathcal{Q}|^2 (1 - \mu)^2], \\
 R_\pm &= \sqrt{\rho^2 + \left(z + \frac{1}{2}R \pm \Sigma\right)^2}, \quad r_\pm = \sqrt{\rho^2 + \left(z - \frac{1}{2}R \pm \sigma\right)^2},
 \end{aligned} \tag{11}$$

where

$$\begin{aligned}
 \mathcal{Q} &= Q + i\mathcal{B}, \quad |\mathcal{Q}|^2 = Q^2 + \mathcal{B}^2, \\
 \Sigma &= \sqrt{M^2 - |\mathcal{Q}|^2 (1 - 2\mu)}, \quad \sigma = \sqrt{m^2 - |\mathcal{Q}|^2 (1 - 2\mu)}, \\
 \mu &= \frac{M + m}{R + M + m}, \quad \kappa = Mm + |\mathcal{Q}|^2 (1 - \mu)^2, \quad \nu = R^2 - M^2 - m^2 + 2|\mathcal{Q}|^2 (1 - \mu)^2,
 \end{aligned} \tag{12}$$

and now the lower dyonic black hole with the horizon  $\rho = 0$ ,  $-\frac{1}{2}R - \Sigma \leq z \leq -\frac{1}{2}R + \Sigma$  has the mass  $M$ , electric charge  $Q$  and magnetic charge  $\mathcal{B}$ , whereas the mass and charges of the upper dyonic constituent with the horizon  $\rho = 0$ ,  $\frac{1}{2}R - \sigma \leq z \leq \frac{1}{2}R + \sigma$  are  $m$ ,  $-Q$  and  $-\mathcal{B}$  (see Fig. 1). Note that in formulae (11) we give the expression of the magnetic potential  $A'_\varphi$  (the imaginary part of the Ernst potential  $\Phi$ ) instead of  $A_\varphi$  because the latter has a rather cumbersome form that can be inferred from Eq. (28) of [3].

There can be no doubt that the spacetime of the asymmetric dyonic dihole (11) is static intrinsically: it is described by a diagonal static metric and, as can be easily checked with the aid of the formula  $\Phi = -A_t + iA'_\varphi$ , the  $\varphi$ -component of the Poynting vector (7) is zero, thus proving the absence of any frame-dragging phenomena in this spacetime.

Let us also mention that the non-equal dyonic black-hole constituents verify the generalized Smarr formula (10). Thus, for instance, restricting ourselves to the lower constituent, we will have

$$\kappa^H = \frac{\Sigma[(R + \Sigma)^2 - \sigma^2]}{(R + M + m)^2 (M + \Sigma)^2}, \quad S^H = \frac{4\pi(R + M + m)^2 (M + \Sigma)^2}{(R + \Sigma)^2 - \sigma^2}, \quad \Phi_{ext}^H = \frac{\mathcal{Q}(1 - 2\mu)}{M + \Sigma}, \tag{13}$$

with which the relation (10) holds identically.

#### IV. CONCLUDING REMARKS

Therefore, our analysis makes it very clear that the dyonic generalizations of the known electrostatic solutions for black diholes [2, 3] are static as well, despite the presence in them of both electric and magnetic charges. The erroneous physical interpretations of the dyonic configurations made in the paper [1] spring from improper use by Cabrera-Munguia et al. of Tomimatsu's formula for the angular momentum [14] that forced those authors to make some incorrect redefinitions (see [15, 16] for details). We hope that our results may be helpful in the future for constructing more general static spacetimes, for instance in the presence of a dilaton field.

#### Acknowledgments

This work was partially supported by Project 128761 from CONACyT of Mexico.

- 
- [1] I. Cabrera-Munguia, C. Lämmerzahl, A. Macías, Phys. Lett. B 743 (2015) 357.
  - [2] R. Emparan, E. Teo, Nucl. Phys. B 610 (2001) 190.
  - [3] V.S. Manko, E. Ruiz, J. Sánchez-Mondragón, Phys. Rev. D 79 (2009) 084024.
  - [4] H. Reissner, Ann. Phys. (Leipzig) 355 (1916) 106.
  - [5] G. Nordström, Proc. K. Ned. Akad. Wet. 20 (1918) 1238.
  - [6] V.S. Manko, J. Martín, E. Ruiz, J. Math. Phys. 35 (1994) 6644.
  - [7] J.A. Cázares, H. García-Compeán, V.S. Manko, Phys. Lett. B 662 (2008) 213; *ibid.* 665 (2008) 426 (Erratum).
  - [8] W.B. Bonnor, Proc. Phys. Soc. Lond. A 67 (1954) 225.
  - [9] F. J. Ernst, Phys. Rev. 168 (1968) 1415.
  - [10] V.S. Manko, R.I. Rabadán, J.D. Sanabria-Gómez, Phys. Rev. D 89 (2014) 064049.
  - [11] L. Herrera, G.A. González, L.A. Pachón, J.A. Rueda, Class. Quantum Grav. 23 (2006) 2395.
  - [12] V.S. Manko, E.D. Rodchenko, B.I. Sadovnikov, J. Sod-Hoffs, Class. Quantum Grav. 23 (2006) 5389.
  - [13] L. Smarr, Phys. Rev. Lett. 30 (1973) 71.
  - [14] A. Tomimatsu, Prog. Theor. Phys. 72 (1984) 73.
  - [15] V.S. Manko, J.D. Sanabria-Gómez, Phys. Rev. D 91 (2015) 088501.
  - [16] V.S. Manko, H. García-Compeán, arXiv:1506.03870 [gr-qc].